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SMOOTH COCYCLES FOR AN IRRATIONAL ROTATION

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ABSTRACT

Explicit examples of smooth cocycles not cohomologous to constants are constructed. Necessary and sufficient conditions on the irrational number θ are given for the existence of such cocycles. It is shown that, depending on θ, the set of C^r cocycles whose skew-product is ergodic is either residual or empty.

1. Introduction

Let G be a countable group which acts on a measure space (X, μ), and let A be a locally compact, second countable, abelian group. A **cocycle** is a Borel map $v : X \times G \to A$ which satisfies

$$v(x, g_1) + v(x \cdot g_1, g_2) = v(x, g_1 + g_2)$$

for all $g_1, g_2 \in G$ and for almost all $x \in X$. v is a **coboundary** if there is a Borel function $w : X \to A$ such that

$$v(x, g) = w(x) - w(x \cdot g)$$

for all $g \in G$ and for almost all $x \in X$. Two cocycles are called **cohomologous** if they differ by a coboundary. We focus our attention on cocycles of an irrational

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rotation. For a fixed irrational θ, we let $X = \mathbb{R}/\mathbb{Z}$ (parameterized by [0,1)), $G = \mathbb{Z}$, $x \cdot n = x + n\theta$, and $A = \mathbb{R}$. In this case, the cocycle v is determined completely by the function $v(x) = v(x,1)$, with $v(x,n) = \sum_{j=0}^{n-1} v(x+j\theta)$ for $n > 0$.

Given a cocycle v, the **skew-product** action of G on $A \times X$ is defined by $(a,x) \cdot g = (a + v(x,g), x \cdot g)$. In the case $G = \mathbb{Z}$, and in particular for the irrational rotation, the skew-product action is generated by the single transformation $T_v(a,x) = (a + v(x,1), x \cdot 1)$. Cohomologous cocycles give rise to isomorphic skew-products, and coboundaries to non-ergodic skew-products [s], though the converses to these statements need not hold.

We address the question of the existence of smooth cocycles which are not cohomologous to a constant, and the related question of the existence of smooth cocycles which give rise to ergodic skew-products. Krygin ([kr]) constructed specific examples of such cocycles for an irrational rotation, with the smoothness of the cocycle depending on how well the irrational θ is approximated by rationals. Nerurkar ([n]) proved that, under conditions on θ similar to those of Krygin, cocycles which give rise to ergodic skew-products form a residual set in the closure of the C^r coboundaries (with C^r metric). In this paper, we slightly relax the conditions on θ and obtain a necessary and sufficient condition for the existence of smooth cocycles for an irrational rotation that give ergodic skew-products. We explicitly construct such cocycles, and show that when they exist they form a residual set.

2.

Our construction will depend on the continued fraction expansion of the irrational number θ. Accordingly, we let

$$\theta = \alpha_0 + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\ddots}}} = [\alpha_0; \alpha_1, \alpha_2, \cdots],$$

with convergents $\frac{m_k}{n_k} = [\alpha_0; \alpha_1, \alpha_2, \cdots \alpha_k]$. The convergents give the best rational approximations to θ relative to size of denominator. The closeness of this approximation is determined by the rate of growth of the denominators of the convergents, which in turn is controlled by the size of the partial quotients $\{\alpha_j\}$. More precisely, if we let $\|x\|$ be the distance from a real number x to the nearest integer, we have:

LEMMA 1:

(i) $\|n_k\theta\| = \min\{\|j\theta\|: 0 < j < n_{k+1}, j \in \mathbb{Z}\}$.

(ii) $\frac{1}{2} < n_{k+1}\|n_k\theta\| < 1$.

(iii) $n_k = \alpha_k n_{k-1} + n_{k-2}$.

(iv) $\alpha_{k+1}\|n_k\theta\| + \|n_{k+1}\theta\| = \|n_{k-1}\theta\|$.

Proof: See [kh]. ∎

Let $\rho = \sup\{\gamma \in \mathbb{Z}: \liminf_{n\to\infty} n^\gamma \|n\theta\| = 0\}$. The **type** of θ is defined similarly, except that the supremum is taken over $\gamma \in \mathbb{R}$ ([kn]). Note that $\rho \geq 0$ for all θ, and that $\rho = 0$ if and only if θ has bounded partial quotients. We will use ρ as a measure of how well θ can be approximated by rationals.

Choose a subsequence of denominators of convergents for θ, $\{n_{l_k}\}$, which satisfy:

(1) $\lim_{k\to\infty} n_{l_k}^\rho \|n_{l_k}\theta\| = 0$

(2) $n_{l_k}\|n_{l_k}\theta\| < \frac{1}{2}$

(3) $n_{l_k} \geq k \max(n_{l_{k-1}+3}, \sum_{j=1}^{k-1} n_{l_j+1})$

The first condition is possible because of the definition of ρ and (i) of Lemma 1. The second condition follows from the first if $\rho \geq 1$; if $\rho = 0$, it is accomplished by shifting l_k by 1 if necessary (see [kh]). The third condition is satisfied by picking a sparse subsequence of those denominators which satisfy the first two. Let $\beta_k = \max\left(\left[\frac{\alpha_{l_k+1}}{2}\right], 1\right)$ (where $[\cdot]$ refers to the greatest integer function). Let $v(x) = \sum_{k=1}^\infty \frac{1}{\beta_k n_{l_k}} e^{2\pi i n_{l_k} x} = f(x) + ig(x)$, and let $v(x,n) = \sum_{m=0}^{n-1} v(x+m\theta) = f(x,n) + ig(x,n)$ be the cocycle determined by v.

LEMMA 2: *There exists a sequence of constants $\{c_j\}$, $\frac{2}{\pi} < |c_j| < \frac{\pi}{2}$, such that*

$$|v(x, \beta_j n_{l_j}) - c_j e^{2\pi i n_{l_j} x}| < 3/j$$

for all x.

Proof: Let $c_j = \frac{(1-e^{2\pi i \beta_j n_{l_j} n_{l_j} \theta})}{\beta_j n_{l_j}(1-e^{2\pi i n_{l_j}\theta})}$. By comparing arclength to chordlength in a circle of circumference 1, we see that for any $x \in [0,1]$,

$$\frac{2\|x\|}{\pi} \leq |1 - e^{2\pi i x}| \leq \|x\|.$$

Also, if $m\|n_{l_j}\theta\| < \frac{1}{2}$, we have $\|mn_{l_j}\theta\| = m\|n_{l_j}\theta\|$. This holds for $m = \beta_j n_{l_j}$, by condition (2) if $\beta_j = 1$ and otherwise by Lemma 1. The desired bounds on c_j follow.

By similar reasoning we see that

$$|v(x,\beta_j n_{l_j}) - c_j e^{2\pi i n_{l_j} x}| \le \sum_{k\neq j} \left| \frac{(1-e^{2\pi i \beta_j n_{l_j} n_{l_k}\theta})}{\beta_k n_{l_k}(1-e^{2\pi i n_{l_k}\theta})} e^{2\pi i n_{l_k} x} \right|$$
$$\le \frac{\pi}{2}\left(\sum_{k=1}^{j-1} \frac{\beta_j \|n_{l_j}\theta\|}{\beta_k \|n_{l_k}\theta\|} + \sum_{k=j+1}^{\infty} \frac{\beta_j n_{l_j}}{\beta_k n_{l_k}}\right).$$

The first sum,

$$\sum_{k=1}^{j-1} \frac{\beta_j \|n_{l_j}\theta\|}{\beta_k \|n_{l_k}\theta\|} \le \sum_{k=1}^{j-1} \frac{n_{l_j}\|n_{l_j-1}\theta\|}{n_{l_k+1}\|n_{l_k}\theta\|}\cdot\frac{n_{l_k+1}}{n_{l_j}}$$
$$< \frac{2}{n_{l_j}}\sum_{k=1}^{j-1} n_{l_k+1}$$
$$< \frac{2}{j}, \text{ by condition (3).}$$

Two applications of the recursion relation (iii) of Lemma 1 show that $\frac{n_{l-2}}{n_l} < \frac{1}{2}$. Thus, by condition (3) again, the second sum,

$$\sum_{k=j+1}^{\infty} \frac{\beta_j n_{l_j}}{\beta_k n_{l_k}} \le \sum_{k=j+1}^{\infty} \frac{n_{l_j+1}}{n_{l_k}}$$
$$\le \frac{1}{j+1}\sum_{k=j+1}^{\infty} (\tfrac{1}{2})^{k-j}$$
$$\le \frac{1}{j+1}. \quad \blacksquare$$

THEOREM 1: *There exists a C^r (periodic) real-valued cocycle for θ, which is not cohomologous to a constant, if and only if* $\liminf_{n\to\infty} n^r\|n\theta\| = 0$.

Proof: Suppose $\liminf_{n\to\infty} n^r\|n\theta\| = 0$, so that $r \le \rho$. Let v be defined as above,

$$v(x) = \sum_{k=1}^{\infty} \frac{1}{\beta_k n_{l_k}} e^{2\pi i n_{l_k} x},$$

and let f be its real part. If we differentiate v term by term, r times, we get the formal power series $\sum_{k=1}^{\infty} \frac{(2\pi i)^r n_{l_k}^{r-1}}{\beta_k} e^{2\pi i n_{l_k} x}$. By condition (1), and using (ii) and (iii) of Lemma 1, we have $\frac{n_{l_k}^r}{\alpha_{l_k+1} n_{l_k}} \to 0$. Thus, by picking a sparser subsequence if necessary, we have l^1 coefficients in the sum above, and thus a continuous r th

derivative for v and therefore f. It remains to show that f is not cohomologous to a constant. Suppose, by way of contradiction, that

$$f(x) = w(x) - w(x+\theta) + K.$$

Then $e^{2\pi i f(x)} = e^{2\pi i K} e^{2\pi i w(x)} / e^{2\pi i w(x+\theta)}$. Hence, because $\|\beta_j n_{l_j} \theta\| \to 0$, there exists a subsequence of the sequence $\{f(\cdot, \beta_j n_{l_j})\}$ that converges in measure to some constant $\lambda = e^{2\pi i K'}$. However, Lemma 2 shows that $|f(x, \beta_j n_{l_j}) - a_j \cos 2\pi n_{l_j} x + b_j \sin 2\pi n_{l_j} x| < \frac{3}{j}$, where $c_j = a_j + i b_j$. The bounds on $|c_j|$ imply that we can find an ϵ such that $\mu(\{x\colon \min_{m \in \mathbb{Z}} |a_j \cos 2\pi n_{l_j} x + b_j \sin 2\pi n_{l_j} x - m - K'| > \epsilon\}) > \frac{1}{2}$ for all j, thus contradicting the convergence assertion above.

Now suppose that $\liminf_{n\to\infty} n^r \|n\theta\| \neq 0$, so that there exists a positive constant C with $n^r \|n\theta\| > C$ for all n. If g is C^r, then its Fourier coefficients, $\{c_n(g)\}$, satisfy $\{n^r c_n(g)\} \in l^2$, and thus $\{\frac{c_n(g)}{1 - e^{2\pi i n\theta}}\} \in l^2$. This shows that the equation $g(x) = w(x) - w(x+\theta) + c_0(g)$ can be solved for w using Fourier series, and thus that g is cohomologous to a constant. ∎

Remark: A consequence of Theorem 1 is that for every irrational number θ there exists a continuous periodic function f, having integral 0, that is not a coboundary for θ. See [kr].

The construction on which Theorem 1 is based can be modified to show the existence of many cocycles which are not coboundaries in C^r, $r \leq \rho$. Further, it can be shown that these functions also have the property that their skew-products are ergodic. The following Theorem shows that this behavior is in fact generic.

Let H_0^r denote the complete metric space consisting of $\{g \in C^r(\mathbb{T})\colon \int g\,du = 0\}$. The C^r metric d_r is defined on C^r by

$$d_r(g,h) = \max_{0 \leq i \leq r} \max_{0 \leq x \leq 1} |g^{(i)}(x) - h^{(i)}(x)|.$$

THEOREM 2: *If $\liminf_{n\to\infty} n^r \|n\theta\| = 0$, then the set S of $g \in H_0^r$ for which T_g is ergodic is residual in H_0^r; if $\liminf_{n\to\infty} n^r \|n\theta\| \neq 0$, the set S is empty.*

Proof: Suppose $\liminf_{n\to\infty} n^r \|n\theta\| = 0$. For each pair (n,k) of positive integers, let $O_{n,k}$ be the set of all $g \in H_0^r$ such that there exists a $j \geq n$ and real numbers a_j and b_j, with $\frac{1}{2} < a_j^2 + b_j^2 < 2$, such that

$$|g(x, \beta_j n_{l_j}) - a_j \cos 2\pi n_{l_j} x - b_j \sin 2\pi n_{l_j} x| < \frac{1}{k},$$

where the sequences $\{n_{l_j}\}$ and $\{\beta_j\}$ are defined as above. Clearly, each $O_{n,k}$ is open in H_0^r. We will prove residuality by first showing that each $O_{n,k}$ is dense in H_0^r, and then showing that if $g \in \cap_k \cap_n O_{n,k}$, then T_g is ergodic.

To prove the density, it will suffice to show that $\overline{O}_{n,k}$ contains the trigonometric polynomials. Accordingly, we let $t(x) = \Re(\sum_{p=1}^m c_p e^{2\pi i p x})$ be an arbitrary trigonometric polynomial. Reasoning as in the proof of Lemma 2, we see that

$$\begin{aligned} |t(x,\beta_j n_{l_j})| &\le \sum_{p=1}^{m} \frac{c_p|1-e^{2\pi i p\beta_j n_{l_j}\theta}|}{|1-e^{2\pi i p\theta}|} \\ &\le \frac{\pi}{2}\sum_{p=1}^{m} \frac{c_p p\beta_j\|n_{l_j}\theta\|}{\|p\theta\|} \\ &\le M\beta_j\|n_{l_j}\theta\| \end{aligned}$$

for some constant M, independent of j. We choose $j_0 > n$ large enough that $M\beta_j\|n_{l_j}\theta\| < \frac{1}{k}$ for $j \ge j_0$. Then, given $\epsilon > 0$, we choose $j > j_0$ large enough that $\frac{(2\pi)^r n_{l_j}^{r-1}}{\beta_j} < \epsilon$. This is possible since by hypothesis $r \le \rho$, so that we have, as in the proof of Theorem 1, $\frac{n_{l_k}^r}{\alpha_{l_k+1} n_{l_k}} \to 0$. If we let $h(x) = \frac{1}{\beta_j n_{l_j}} e^{2\pi i n_{l_j} x}$, then this last restriction on j guarantees that $\|h\|_{C^r} = d_r(h,0) < \epsilon$. If we let a_j and b_j be defined as in Theorem 1, so that $a_j\cos 2\pi n_{l_j}x + b_j \sin 2\pi n_{l_j} x = \Re(h(x,\beta_j n_{l_j}))$, the condition $j > j_0$ finishes the proof of density by showing that $t+\Re(h) \in O_{n,k}$.

Now we let $g \in \cap_k \cap_n O_{n,k}$, and show the ergodicity of T_g. Let $E(g)$ denote the group of essential values of g (See [s]). It will suffice to show that $E(g) \neq \lambda\mathbb{Z}$ for any $\lambda \ge 0$.

Suppose $E(g) = \lambda\mathbb{Z}$. We will make use of a result of K. Schmidt, which implies that for any compact set K with $K \cap E(g) = \emptyset$, there is a Borel set B, $\mu(B) > 0$, such that $B \cap B - \beta_j n_{l_j}\theta \cap \{x\colon g(x,\beta_j n_{l_j}) \in K\} = \emptyset$ for all $j > 0$ (Prop. 3.8 of [s]). If $\lambda \neq 0$, for each $\epsilon > 0$, we define the compact set $K_\epsilon = \{x \in [-2,2]\colon |x - \lambda\mathbb{Z}| \ge \epsilon\}$; if $\lambda = 0$, we let K_ϵ be defined as if $\lambda = 1$.

An elementary estimate produces an $\epsilon_0 > 0$ such that $\mu(\{x \in [0,1)\colon a\cos 2\pi x + b\sin 2\pi x \in K_{\epsilon_0}\}) > \frac{1}{2}$ for any $a, b \in \mathbb{R}$ with $\frac{1}{2} < |a|^2 + |b|^2 < 2$. This gives, for each j and for each interval I of length a multiple of $\frac{1}{n_{l_j}}$,

$$\mu(\{x \in I\colon a_j \cos 2\pi n_{l_j} x + b_j \sin 2\pi n_{l_j} x \in K_{\epsilon_0}\}) > \frac{\mu(I)}{2},$$

where a_j and b_j are as in the definition of $O_{n,k}$. We let $K = K_{\frac{\epsilon_0}{2}}$, and obtain for each n, an integer $j > n$ such that $\mu([y, y+\frac{6}{n_{l_j}}) \cap \{x\colon g(x,\beta_j n_{l_j}) \in K\}) > \frac{3}{n_{l_j}}$, for all y.

We will reach a contradiction, via Schmidt's result, by showing that for any Borel set B with $\mu(B) > 0$, and for j sufficiently large, there exists a $y \in [0,1)$ with $\mu(B \cap (B - \beta_j n_{l_j}\theta) \cap [y, y + \frac{6}{n_{l_j}})) > \frac{3}{n_{l_j}}$. Thus, let B be a fixed set of positive measure. Since almost every point of B is a point of density, we have that, for almost all $y \in B$, $\mu(B \cap [y, y + \frac{6}{n_{l_j}}) > \frac{5}{n_{l_j}}$ for j sufficiently large. Because $\|\beta_j n_{l_j}\theta\| < \|n_{l_j-1}\theta\| < \frac{1}{n_{l_j}}$, for such a y and j we also have that $\mu(B - \beta_j n_{l_j}\theta) \cap [y, y + \frac{6}{n_{l_j}})) > \frac{4}{n_{l_j}}$. It follows that $\mu(B \cap (B - n_{l_j}\theta) \cap [y, y + \frac{6}{n_{l_j}})) > \frac{3}{n_{l_j}}$, and we have established the ergodicity of T_g.

To prove the second statement, suppose T_g is ergodic, and $g \in C^r$. Then g cannot be a coboundary and must have integral 0 [s]. Thus by Theorem 1, $\liminf_{n\to\infty} n^r\|n\theta\| = 0$. ∎

Remark: Theorem 2.2 of [n] implies that the set S is residual in H_0^r under the more stringent assumption that $\liminf_{n\to\infty} n^{r+\epsilon}\|n\theta\|$ is finite for some $\epsilon > 0$.

If g is a coboundary for θ, i.e., $g(x) = w(x) - w(x+\theta)$, then the function $u(x,y) = y + w(x)$ is a nonconstant invariant function for the skew-product T_g. Hence, we have the following immediate corollary of Theorems 1 and 2.

COROLLARY: *If* $\liminf_{n\to\infty} n^r\|n\theta\| = 0$, *then the set of* C^r *functions that are coboundaries for* θ *is of the first category in* H_0^r; *if* $\liminf_{n\to\infty} n^r\|n\theta\| \neq 0$, *the set of* C^r *functions that are coboundaries for* θ *coincides with* H_0^r.

Remark: The residuality result of Theorem 2 provides an existence proof for continuous periodic functions whose associated skew-products are ergodic. We emphasize that the function f constructed in the proof of Theorem 1,

$$f(x) = \Re\Big(\sum_{k=1}^{\infty} \frac{1}{\beta_k n_{l_k}} e^{2\pi i n_{l_k} x}\Big),$$

is an explicit example of such a function.

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